AGT conjecture and convolution algebras

Hiraku Nakajima RIMS, Kyoto University

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Osaka City University

AGT conjecture

Nekrasov deformed partition function for $N=2$ SUSY (pure) YM theory on $\mathbb{R}^4$

$$Z(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = (v|v)$$

v: Whittaker vector in Verma module of the W-algebra

mathematical formulation

G : compact Lie group

$M(G,n)$ = moduli space of G-instantons on $S^4 = \mathbb{R}^4 \cup \infty$ $(\mathbb{R}^4 = \mathbb{C}^2)$
with framing at ∞, "instanton number" $= n$

$$\curvearrowleft \mathbb{T} := \underbrace{U(1)\times U(1)}_{\text{base}} \times \underbrace{T}_{\text{framing}} \subset U(2)\times G$$

Conjecture. $\bigoplus_n H^*_{\mathbb{T}}(M(G,n))$ has a structure of the (dual) Verma module of the W-algebra

equivariant cohomology, precise later

such that $\sum$ [the fundamental classes] $= v$

By the convolution product, $\bigoplus_n H^*_{\mathbb{T}}(M(G,n))$ is a module of the convolution algebra.

This is a general and abstract construction, which has been used many years with success in geometric representation theory.
(Kazhdan-Lusztig, Ginzburg, Vasserot, Schiffmann, N, ...)

But in general, it is difficult to determine the convolution algebra explicitly.

In the previous examples, the convolution algebras were identified with more concrete algebras given by generators and relations
(Weyl groups, Hecke algebras, Kac-Moody Lie algebras, etc.)
often by lengthy computation.

W-algebras do not have such presentation in general.
$\longrightarrow$ More conceptual understanding will be required.

So the surprising point of the AGT conjecture is the identification of the convolution algebra with the W-algebra.

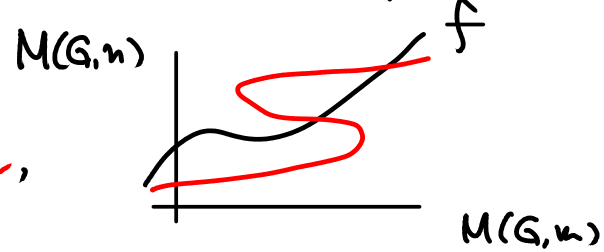
Convolution algebra (not rigorous)

K : cohomology class on a product $M(G,n)\times M(G,m)$ gives an operator by

$$(\alpha|K|\beta) = \int_{\substack{M(G,n)\times M(G,m)\\ x \qquad\quad y}} \alpha(x)\,K(x,y)\,\beta(y)$$

$$\alpha \in H^*_{\mathbb{T}}(M(G,n)),\quad \beta\in H^*_{\mathbb{T}}(M(G,m))$$

- One of typical examples of K is the Poincaré dual of the graph of a map $f: M(G,m)\to M(G,n)$.



But K is more general, can be multi-valued, etc.

- We require $(\alpha|K|\beta)$ to be well-defined. This is nontrivial, as $M(G,n)\times M(G,m)$ is noncompact and the integral a priori may diverge.

Convolution algebra = the algebra of all such operators

Conjecture is proved when the gauge group $= SU(N)$ (or $U(N)$)

by $\begin{cases} \text{Maulik–Okounkov} \\ \text{Schiffmann–Vasserot} \end{cases}$

Today : Explain their proof in a formulation working for general G

$\longrightarrow$ a step towards a proof for general G

Key ingredient : Relate $\bigoplus_n H^*_{\mathbb{T}}(M(G,n))$ and $\bigoplus_{n'} H^*_{\mathbb{T}}(M(L,n'))$ where

$L \subset G$ is a Levi subgroup.

e.g. $G = SU(N) \supset S(U(N_1)\times\cdots\times U(N_k))$ $\begin{bmatrix} \boxed{N_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{N_k} \end{bmatrix}$

$N_1+\cdots+N_k = N$

▷ For $L=T$, this gives $W_G \xrightarrow{\cong} \bigcap \operatorname{Ker}(\text{screening op.}) \subset \text{Heisenberg}^{\otimes \text{rank}}$

▷ In general, a generalised Miura transform $W_G \longrightarrow W_L$.

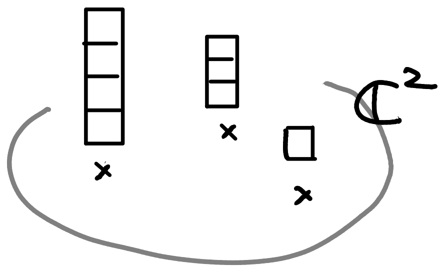
<u>1st step</u> $G = U(1)$ (We should get $W_G = \text{Heis.}$)

Naively $M(U(1), n) = \phi$ unless $n = 0$

However $\overline{M}(U(1), n) \equiv$ ADHM moduli space

$= S^n(\mathbb{C}^2)$: n-th symmetric product of $\mathbb{C}^2$

$= \coprod_{\lambda \vdash n} S^n_\lambda \mathbb{C}^2$ ← configuration of points



So define

$$H^*_{\mathbb{T}}(M(U(1), n)) := H^*_{\mathbb{T}, \text{orb}}(\overline{M}(U(1), n)) \qquad \mathbb{T} = \underset{\varepsilon_1,}{U(1)} \times \underset{\varepsilon_2}{U(1)} \times \underset{a_0}{U(1)}$$

$$= \bigoplus_{\lambda \vdash n} H^*_{\mathbb{T}}(\overline{S^n_\lambda \mathbb{C}^2})$$

An equivariant cohomology is a module over $H^*_{\mathbb{T}}(pt) = H^*(B\mathbb{T}) = \mathbb{C}[\varepsilon_1, \varepsilon_2, a_0]$

<u>Th</u> (cf. N, Grojnowski 1995)

$\bigoplus_n H^*_{\mathbb{T}}(M(U(1), n))$ has a structure of the Fock space :

$$[a_i, a_j] = (-1)^{i-1}\, i\, \delta_{i+j,0}$$

<u>Idea</u> $S^n\mathbb{C}^2 \times S^m\mathbb{C}^2 \longrightarrow S^{n+m}\mathbb{C}^2$

$$H^*_{\mathbb{T}}(\overline{S^n_\lambda \mathbb{C}^2}) \otimes H^*_{\mathbb{T}}(\overline{S^m_\mu \mathbb{C}^2}) \longrightarrow H^*_{\mathbb{T}}(\overline{S^{n+m}_{\lambda\sqcup\mu}}\mathbb{C}^2)$$

This gives a multiplication on the Fock space

$$\mathrm{Fock} \cong \mathbb{C}[a_{-1}, a_{-2}, \cdots] \otimes \mathbb{C}[\varepsilon_1, \varepsilon_2, a_0]$$

$$\underset{(i>0)}{a_i} = \varepsilon_1\varepsilon_2 \frac{\partial}{\partial a_{-i}} = \varepsilon_1\varepsilon_2 \times (\text{transpose of } a_{-i} \text{ w.r.t. } (\ |\ ))$$

$$\text{fundamental class} = \sum_n \frac{1}{n!} a_1^n = \exp(a_1)$$

$$\Longrightarrow \quad (v|v) = (\exp(a_1)|\exp(a_1)) = \exp\left(\frac{1}{\varepsilon_1\varepsilon_2}\right)$$

<u>Th</u> (Lehn 1998)
A certain natural operator in the convolution algebra gives the Virasoro algebra.

<u>2^{nd}</u> G : almost simple e.g. $SU(N)$, $Spin(N)$, E_8

$M(G,n)$ is not correct space.

$\overline{M}(G,n) = \coprod_{m=0}^{n} M(G,n-m) \times S^m\mathbb{R}^4$: Uhlenbeck partial cptification

$H^*_{\mathbb{I}}(M(G,n)) := IH^*_{\mathbb{I}}(\overline{M}(G,n))$: intersection cohomology group

<u>Rem.</u> $IH^*_{\mathbb{I}}(\overline{M}(G,n))$ satisfies the Poincaré duality though $\overline{M}(G,n)$ is singular.

$\Longrightarrow$ appropriate to consider convolution products.

More generally if $G \cong G_1 \times G_2 \times \cdots \times G_k$ (almost product)

$$H^*_{\mathbb{I}}(M(G,n)) := \bigoplus_{n_1+\cdots+n_k=n} H^*_{\mathbb{I}}(M(G_1,n_1)) \otimes \cdots \otimes H^*_{\mathbb{I}}(M(G_k,n_k))$$

3rd geometric realization of Miura transforms

$$L_{\mathbb{C}} \subset \underset{\text{parabolic}}{P} \subset G_{\mathbb{C}} \qquad \text{e.g. } P = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$$

<u>Conjecture'</u> There exists a natural homomorphism (depending on P)

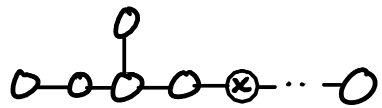
$$H^*_{\mathbb{T}}(M(L,n)) \longrightarrow H^*_{\mathbb{T}}(M(G,n))$$

given by a cohomology class K on $M(L,n) \times M(G,n)$,

intertwines the module structures of convolution algebras.

It becomes $\cong$ after $\otimes_{H^*_{\mathbb{T}}(pt)}$ (quotient field).

Conj.' + (Conj. for $\underset{\uparrow\,\text{ok}}{G = SU(2)}$) $\Longrightarrow$ Conj.

(☺ Take $L =$ 

$$\mathrm{Conv}_G \subset \mathrm{Conv}_L \subset \mathrm{Conv}_T$$

$$\mathrm{Conv}_L = \mathrm{Heis}^{\otimes i} \otimes \mathrm{Vir} \otimes \mathrm{Heis}^{\otimes r-i-1} \subset (\mathrm{Heis.})^{\otimes r} = \mathrm{Conv}_T$$

The construction of K is the key.
The graph of $M(L,n) \hookrightarrow M(G,n)$ is not correct.

For $G = SU(N)$, K is very roughly given as follows:

$$M(G,n) = \{\text{holomorphic vector bundles } E \text{ on } \mathbb{CP}^2 = \mathbb{C}^2 \cup \ell_\infty \text{ with framing} \quad \operatorname{rk} E = N,\ c_2 = n\}$$

$$M(L,n) = \{N\text{-tuples of holo. line bundles } L_1 \oplus \cdots \oplus L_N\}$$

$$\mathfrak{T} := \{(E, L_1 \oplus \cdots \oplus L_N) \mid E \text{ has a filtration } E = E^N \supset E^{N-1} \supset \cdots \supset E^0 = \{0\} \text{ s.t. } E^i/E^{i-1} \cong L_i \quad (i = 1, \dots, N)\}$$

with appropriate multiplicities.

$\Longrightarrow K =$ Poincaré dual of $\mathfrak{T}$